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Electrostatic forces on a conducting sphere due to a charged, insulating plane

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Abstract. The electrostatic force on a conducting sphere positioned symmetrically above a uniformly charged, insulating, circular disc is calculated using the method of images. Results for spheres which are either earthed, uncharged or carry a known charge are given in terms of infinite series which converge rapidly under certain stated conditions. The dependence of the force on sphere radius, sphere–plane separation and the radius of the charged area is presented graphically.

1. Introduction

Recently there has been considerable interest in the nature of the forces responsible for adhesion at interfaces (Krupp 1967, van den Temple 1972). At a particular interface these forces may be due to a variety of phenomena which fall into three categories. The strongest attractive interactions result from metallic, covalent or ionic chemical bonds. These forces are essentially short ranged. Weaker, long-range interactions are caused by van der Waals or dispersion forces and electrostatic forces. Electrostatic attractions may be due to charge transfer between bodies of different work function giving rise to a double layer of charge at their interface. Alternatively, one of the adherents may carry an excess of electric charge on or near to its surface. It has been shown (Krupp 1967) that, in general, dispersion forces will predominate over electrostatic forces. However, for metals in contact with non-metals, high densities of charge on the non-metal surface may give rise to attractive electrostatic forces considerably in excess of any dispersion forces. Such a case has recently been observed by Higginbotham *et al* (1975) in a study of adhesion at the gold–mica interface. It is also well known that a tribo-electric charge on plastic can attract small metal spheres with forces greater than their own weight.

One experimental method frequently employed in the measurement of adhesive forces is to find the force required to remove a metal sphere or hemispherical tip from a plane surface, either by means of an ultracentrifuge (Bohme *et al* 1965), a gravity technique (Howe *et al* 1955) or a microbalance (Kohno and Hyodo 1974). Theories of adhesion due to dispersion forces (Craig 1973) and electrostatic double layers (Derjaguin and Smilga 1967) are well developed and expressions for attractive forces between sphere and half-sphere due to each of these phenomena have been given by Krupp and Sperlung (1966). However, the literature contains no theory of electrostatic attraction for the case of a non-equilibrium excess charge residing on an insulating surface.

2. Theory

The theory given below examines the situation where an earthed conducting sphere in vacuum is positioned symmetrically above a circular area of charge density σ bound to a thin insulating sheet. The assumptions made in the solution of the problem are as follows.

(i) The insulating sheet has no dimension perpendicular to the plane of the disc. Thus, the bound charges which would normally be induced on the front and rear surfaces of the insulator must be considered to be coincident, so that they will have no net effect. Since the insulating disc is treated as two dimensional, it will not give rise to any image charges. The implications of this assumption for the relationship between this theory and the experimentally encountered situation of an insulator of finite thickness are discussed in § 6.

(ii) The sphere and plane have idealized, smooth surfaces.

(iii) A uniform charge density σ resides within a circular disc on the insulating sheet. Outside the disc the surface charge is zero. The charge is not mobile so that σ is unaffected by the presence of the conducting sphere.

(iv) The sphere is earthed.

The steps in the solution of the problem are:

(*a*) Calculation of the image charge. For the purposes of calculating the attractive force on the sphere, the sphere may be replaced by a surface charge density σ', which is the image of σ in the sphere. σ' lies on an imaginary surface within the sphere.

(*b*) Derivation of expressions for the potentials Φ_p (for the plane) and Φ_s (for the sphere) at a general point due to the charge distributions σ and σ' respectively.

(*c*) Determination of the total potential $\Phi = \Phi_p + \Phi_s$.

(*d*) Derivation of the true surface charge density σ_s on the sphere.

(*e*) Calculation of the total attractive force on the sphere.

The coordinate system used to specify the problem is shown in figure 1. The sphere of radius a has its centre a distance p from the plane. The radius of the charged disc is c.

2.1. *The image charge density* σ'

A charge q a distance d from the centre of an earthed sphere of radius a has an image charge $q' = -qa/d$ located a distance a^2/d from the centre of the sphere on a line joining q to the centre (see eg Lorrain and Corson 1970, p 147). If the locus of d is such that q is spread over a plane whose minimum separation from the centre of the sphere is p, then the locus of q' is a sphere of radius $b = a^2/2p$, as shown in figure 1. Considering the relative sizes of corresponding elementary areas dS_1 and dS_2 on the plane and the image sphere, the charge density on the image sphere can be shown to be

$$\sigma' = -\frac{p^3\sigma}{a^3\cos^3\theta}.$$

As $\theta \to \pi/2$ the area of the plane being imaged increases so that $\sigma' \to \infty$. It is therefore necessary that the charged area of the plane must be limited to a circle of finite radius c, so that the image charges are confined to that segment of the image sphere for which $\theta < \theta_0$.

2.2.1. The potential due to σ. The potential at a point P_0 lying on the polar axis OZ,

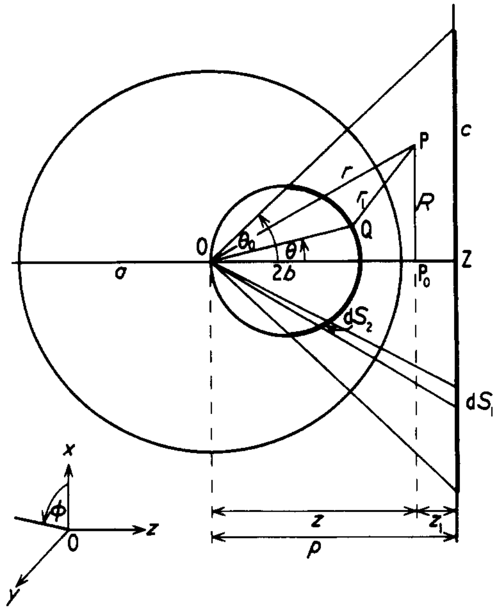

Figure 1. The Cartesian (x, y, z) and spherical polar (r, θ, ϕ) coordinate systems employed in the analysis. The insulating plane carries a uniform surface charge density σ within a disc of radius c. The earthed conducting sphere has radius a and the image sphere has radius b.

due to the charge density σ, is

$$\Phi_p^a = \frac{\sigma}{2\epsilon_0}[(c^2+z_1^2)^{1/2} - z_1].$$

Putting $z_1 = p - z$ (see figure 1), this can be expressed in the form

$$\Phi_p^a = \frac{\sigma}{2\epsilon_0}\left[(c^2+p^2)^{1/2}\left(1 - \frac{z(2p-z)}{c^2+p^2}\right)^{1/2} - p + z\right].$$

Since $z(2p-z) \leqslant c^2+p^2$, the term in large parentheses may be expanded as an infinite series to give an expression of the form

$$\Phi_p^a = \frac{\sigma}{2\epsilon_0}\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

where

$$b_0 = (c^2+p^2)^{1/2} - p$$

$$b_1 = 1 - \frac{p}{(c^2+p^2)^{1/2}}$$

$$b_2 = \frac{1}{2(c^2+p^2)^{1/2}} - \frac{p^2}{2(c^2+p^2)^{3/2}}$$

$$b_3 = \frac{p}{2(c^2+p^2)^{3/2}} - \frac{p^3}{2(c^2+p^2)^{5/2}}$$

and

$$b_n = (c^2+p^2)^{1/2} \sum_k \frac{(2k-3)!(-1)^{n-k+1} p^{2k-n}}{2^{n-2}(k-2)!(n-k)!(2k-n)!(c^2+p^2)^k}$$

where the limits of k in the summation are $n/2$ to n if n is even and $(n+1)/2$ to n if n is odd.

Thus, the potential at a general point P in the region $2b < r < p$, due to the charge density σ on the disc of radius c is

$$\Phi_p = \frac{\sigma}{2\epsilon_0}\left(\sum_{n=0}^{\infty} b_n r^n P_n(\cos\theta)\right)$$

where $P_n(\cos\theta)$ are the Legendre polynomials.

2.2.2. The potential due to σ'. The potential at a general point P due to a charge $\sigma' dS_2$ at Q (see figure 1) is given by

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{\sigma' \, dS_2}{r_1}.$$

Therefore the potential at P due to the total image charge is

$$\Phi_s = -\frac{1}{4\pi\epsilon_0}\int_0^{\theta_0}\int_0^{2\pi} \frac{p^3\sigma}{a^3\cos^3\theta}\frac{a^4}{p^2}\frac{\sin\theta\cos\theta}{r_1}\, d\theta\, d\phi.$$

Provided Q is allowed to take on all values of ϕ, no generality is lost by constraining P to lie in the plane $y = 0$. Then

$$\mathbf{OP} = R\boldsymbol{i} + z\boldsymbol{k}$$

$$\mathbf{OQ} = 2\,B\sin\theta\cos\theta\cos\phi\,\boldsymbol{i} + 2b\sin\theta\cos\theta\sin\phi\,\boldsymbol{j} + 2b\cos^2\theta\,\boldsymbol{k}.$$

Therefore

$$\Phi_s = -\frac{pa\sigma}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{\sin\theta}{\cos^2\theta}\left(\int_0^{2\pi}\frac{d\phi}{[R^2+z^2+4b(b-z)\cos^2\theta - 4bR\sin\theta\cos\theta\cos\phi]^{1/2}}\right) d\theta.$$

Since it is necessary to find the gradient of this potential, the integral must be determined in analytical form. This can be done if R is set equal to zero, ie if the point P lies on the polar axis OZ. The integral then becomes

$$\int_0^{\theta_0}\frac{2\pi\sin\theta\, d\theta}{z\cos^2\theta\{1+[4b(b-z)\cos^2\theta/z^2]\}^{1/2}}$$
$$= \frac{2\pi}{z^2}\left(\frac{1}{\cos\theta_0}[(z-2b\cos^2\theta_0)^2 + 4b^2\sin^2\theta_0\cos^2\theta_0]^{1/2} - (z-2b)\right).$$

Thus

$$\Phi_s^a = -\frac{\sigma pa}{2\epsilon_0 z}\left\{\frac{1}{\cos\theta_0}\left[1+\frac{4b\cos^2\theta_0}{z}\left(\frac{b}{z}-1\right)\right]^{1/2} - \left(1-\frac{2b}{z}\right)\right\}.$$

Since all points P of interest lie outside the conducting sphere, then $b/z \leqslant \frac{1}{2}$ and the term in square brackets may be expanded as an infinite series to give an expression of the form

$$\Phi_s^a = -\frac{\sigma p a}{2\epsilon_0 z}\left[\sum_{n=0}^{\infty} a_n\left(\frac{b}{z}\right)^n\right]$$

where

$$a_0 = \frac{1}{\cos\theta_0} - 1$$

$$a_1 = 2 - 2\cos\theta_0$$

$$a_2 = 2\cos\theta_0 - 2\cos^3\theta_0$$

$$a_3 = 4\cos^3\theta_0 - 4\cos^5\theta_0$$

and

$$a_n = \sum_k \frac{4(2k-3)!(-1)^{n-k+1}\cos^{2k-1}\theta_0}{(k-2)!(n-k)!(2k-n)!}$$

where the limits of k in the summation are $n/2$ to n if n is even and $(n+1)/2$ to n if n is odd. Thus, the potential at a general point P due to the charge density σ' on the segment $\theta \leqslant \theta_0$ of the image sphere is

$$\Phi_s = -\frac{\sigma p a}{2\epsilon_0}\left(\frac{a_0}{r} + \sum_{n=1}^{\infty}\frac{a_n}{r^{n+1}} b^n \mathrm{P}_n(\cos\theta)\right).$$

2.3. *The total potential*

Recalling that $b = a^2/2p$ and that $\cos\theta_0 = p/(c^2+p^2)^{1/2}$ it is observed that the coefficients a_n and b_n are related by the expression

$$b_n = \frac{1}{2^n p^{n-1}} a_n.$$

The total potential $\Phi = \Phi_p + \Phi_s$ in the region outside the sphere can therefore be written

$$\Phi = \frac{\sigma p}{2\epsilon_0}\left(\sum_{n=0}^{\infty}\frac{a_n}{2^n p^n} r^n \mathrm{P}_n(\cos\theta)\right) - \frac{\sigma p a}{2\epsilon_0}\left(\frac{a_0}{r} + \sum_{n=1}^{\infty}\frac{a_n}{r^{n+1}}\frac{a^{2n}}{2^n p^n}\mathrm{P}_n(\cos\theta)\right).$$

The emergence of a simple relationship between a_n and b_n is to be expected since on the surface of the sphere Φ must be zero. Cancellation of all terms in two infinite series is unlikely to occur unless each term in one series cancels with an equivalent term in the other. As a check on the expressions for Φ_p and Φ_s, it is found that substituting $r = a$ in the above expression yields $\Phi = 0$.

2.4. *The charge density on the surface of the sphere*

The field at the surface of the sphere may be found by differentiation of the total potential Φ with respect to r at $r = a$:

$$E = \frac{\sigma}{2\epsilon_0}\sum_{n=0}^{\infty}\frac{2n+1}{2^n}\left(\frac{a}{p}\right)^{n-1} a_n \mathrm{P}_n(\cos\theta).$$

The charge density σ_s on the sphere can then be written

$$\sigma_s = \tfrac{1}{2}\sigma \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$$

where

$$c_n = \left(\frac{a}{p}\right)^{n-1} \frac{2n+1}{2^n} a_n$$

and the a_n are as defined in § 2.2.2.

2.5. The attractive force on the sphere

The force on the sphere towards the plane is given by

$$F_e = \iint_{\text{sphere}} \frac{\sigma_s^2}{2\epsilon_0} \cos\theta \, dS = \frac{\pi\sigma^2 a^2}{4\epsilon_0} I$$

where

$$I = \int_{-1}^{1} \mu \left(\sum_{n=0}^{\infty} c_n P_n(\mu) \right)^2 d\mu.$$

Using the recurrence relations and the orthogonality properties of the Legendre polynomials, this reduces to

$$I = 2 \sum_{n=0}^{\infty} \int_{-1}^{1} c_n c_{n+1} P_n(\mu) P_{n+1}(\mu) \, \mu \, d\mu.$$

Therefore

$$I = 2 \sum_{n=0}^{\infty} c_n c_{n+1} \frac{n+1}{2n+3} \frac{2}{2n+1}.$$

The convergence of this series is sufficiently rapid for a reliable value to be computed from the first twenty terms. Additional terms were found to cause no changes greater than 1 part in 10^4.

For given values of a, p and c, I can be evaluated, and the force F_e on the sphere can then be calculated for any particular value of the charge density σ.

3. Convergence of the series

A sufficient condition for the convergence of the series is that

$$\sum_{n=0}^{\infty} a_n \left(\frac{b}{z}\right)^n$$

in the expression for Φ_s^a be absolutely convergent. This series is composed of terms of the form

$$\frac{(-1)(2i-3)!4^i \cos^{2i-1}\theta_0}{2^{2i-2}(i-2)!} \left(\frac{b}{z}\right)^i \sum_{m=0}^{i} \frac{(-1)^m}{m!(i-m)!} \left(\frac{b}{z}\right)^m.$$

The condition for the absolute convergence of the series of these terms from $i = 2$ to ∞ can be shown to be

$$c^2 > 2a^2 - (p-a)^2.$$

When the sphere touches the plane, $p = a$, and the condition for convergence is $c > a\sqrt{2}$.

The condition that the series be absolutely convergent is in fact overstrict. It would appear from an inspection of the terms in the summation that the series will converge provided $c > a$.

4. Graphical presentation of results

Figure 2 shows the dependence of the force F_e on sphere radius a for the case where the sphere touches the plane ($p - a = 0$). The graph was extended to values of $a > c$ by calculating the asymptotic value of F_e as a tends to infinity. Provided $c \gg p - a$, the problem then reduces to that of a parallel plate condenser where plates of radius c carry charge densities $\pm\sigma$.

Force/K is plotted as ordinate where $K = \pi\sigma^2/4\epsilon_0$. The asymptotic value is then $2c^2$. F_e/K is plotted only for $c = 1$. To find F_e/K for $a = a_x$ and $c = c_x$, the value of F_e/K is read at $a = a/c = a_x/c_x$ and multiplied by $(c_x/c)^2 = c_x^2$. Graphs of F_e/K against a for $p - a > 0$ show a similar overall shape, but must be calculated separately for each individual set of values of a, c and p.

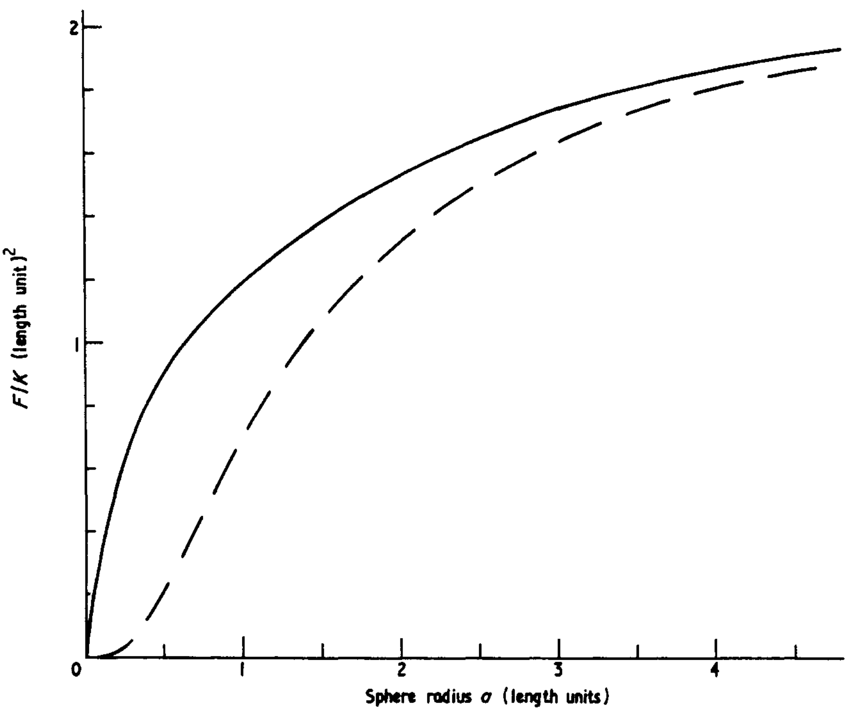

Figure 2. The forces on earthed (full curve) and uncharged (broken curve) spheres as functions of sphere radius a for the case where the sphere touches the plane. The curve shown is for disc radius $c = 1$ but may be scaled to any case of similar geometry as described in the text.

Figure 3 shows the dependence of F_e/K on disc radius c for the case $p-a=0$. Figure 4 shows how F_e/K varies with sphere–plane separation $p-a$ for the case $c=1$.

5. Forces on charged and uncharged spheres

The above theory for an earthed sphere is easily modified for the case of an uncharged sphere. For an earthed sphere, the total image charge q'_t is given by

$$q'_t = \int_0^{\theta_0} 2\pi\sigma' \frac{a^4}{p^2} \sin\theta\cos\theta\, d\theta = 2\pi\, pa\,\sigma\left(1 - \frac{1}{\cos\theta_0}\right).$$

If the sphere is uncharged, the equivalent charge system when the sphere is removed consists of the image charge q'_t as for the case with the sphere earthed, and an additional point charge $-q'_t$ at point O. This gives rise to one extra term in the series for I which cancels with the first term $n=0$. Thus the force on an uncharged sphere is

$$F_u = \frac{\pi\sigma^2 a^2}{4\epsilon_0} I_u$$

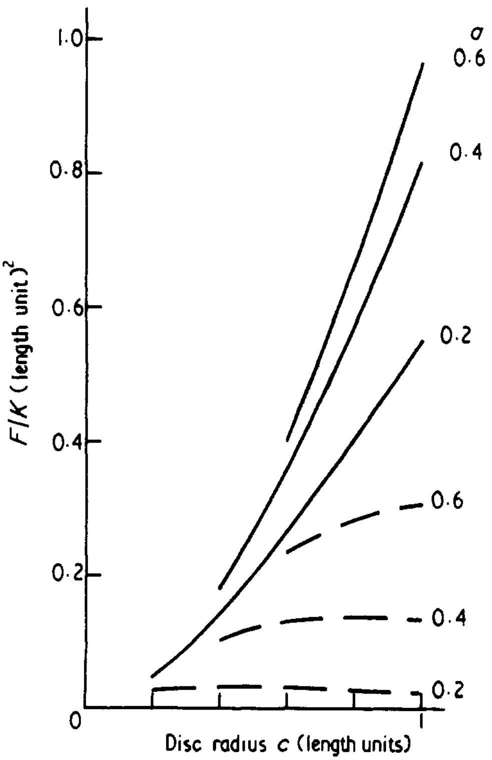

Figure 3. The forces on earthed (full curves) and uncharged (broken curves) spheres as functions of disc radius c for the case where the sphere touches the plane.

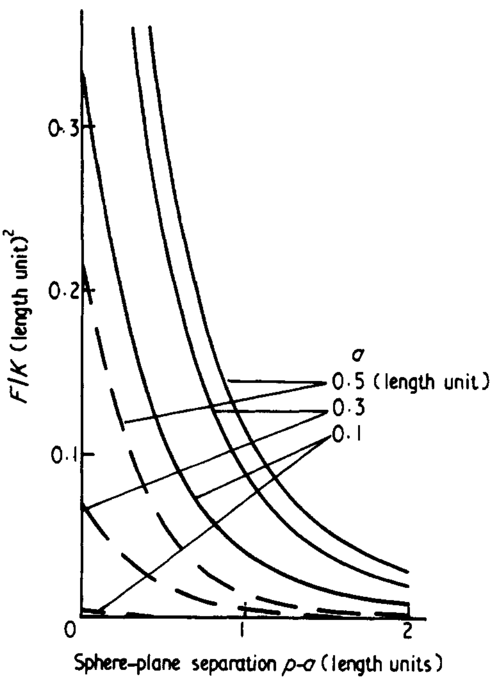

Figure 4. The forces on earthed and uncharged spheres as functions of sphere–plane separation $p-a$.

where

$$I_u = 2 \sum_{n=1}^{\infty} c_n c_{n+1} \frac{n+1}{2n+3} \frac{2}{2n+1}.$$

The dependence of F_u/K on a, c and $p-a$ is shown by the broken curves in figures 2, 3 and 4.

The force on a sphere carrying a charge Q can be found by placing a further additional charge Q at point O. The force is then given by

$$F_c = \iint_{\text{sphere}} \frac{1}{2\epsilon_0}\left(\sigma_s + \frac{Q-q_t'}{4\pi a^2}\right)^2 \cos\theta \, dS.$$

Again, this introduces one extra term into the expression for the force to give

$$F_c = \frac{\pi\sigma^2}{4\epsilon_0}\left(a^2 I_u + \frac{2Q}{3\pi\sigma}c_1\right).$$

6. Discussion

The assumption which most seriously limits the applicability of the theory is that the insulating material is considered to have infinitesimal thickness. Only for this case can the problem be solved in terms of the one image charge. As soon as the insulator is allowed to become even one atom thick, the polarization of the atoms will be influenced by the presence of the sphere. If the problem is to be solved in terms of image charges, a double infinity of images will then be required, one series within the sphere and the other behind the front surface of the insulator. In the most interesting region where the sphere is close to or in contact with the plane, many images of complicated shape and charge distribution must be considered, and the problem becomes intractable. As the sphere recedes from the plane, the images approximate to point charges at the centre of the sphere and at the mirror point behind the insulator surface. These two infinite series of charges may then be summed, and the force between them calculated. Provided $p \gg a$, an estimate can be made of the percentage difference between the forces calculated above and those to be expected in the case of an insulator of finite thickness. As the sphere approaches the plane ($p \gtrsim a$) the approximation that the image charges are at the centre of the sphere is not realistic. In this case, an overestimate of the percentage difference may be found by assuming that the same image charges are located at points on the z axis, distance $\pm(p-2b)$ from the charged surface. On this basis the forces in figure 4 can be shown to apply also in the case of a thick insulator of dielectric constant 3 within the following limits: (i) the forces shown form a lower bound; (ii) they are accurate within 2% at $a = 0{\cdot}1$, $p-a = 2$ and 15% at $a = 0{\cdot}5$, $p-a = 2$; (iii) the accuracy decreases as $p-a$ decreases (eg 7% at $a = 0{\cdot}1$, $p-a = 1$ and 55% at $a = 0{\cdot}5$, $p-a = 1$).

These arguments cannot be applied to figures 2 and 3 where $p-a = 0$. However, in figure 2 it is known that the graph passes through the origin, and the general shape of the curve for a thick insulator will be as shown with an error at each point of less than $[\frac{1}{2}(\epsilon_r + 1)]^2 F/K$ due to the bound surface charge (see Lorrain and Corson 1970, p 153).

A more accurate analysis of the problem of a metallic sphere close to a charged insulator of finite thickness in terms of a numerical solution of Laplace's equation will be the subject of a subsequent paper.

7. Conclusion

The electrostatic forces on earthed conducting spheres due to a uniform distribution of charge on the surface of an insulating disc have been calculated. The analysis has been extended both to uncharged spheres and to spheres carrying a known charge. Forces are expressed as infinite series which are rapidly convergent subject to given conditions. Results are also presented graphically. Where possible, graphs have been given in a form which allows the force to be scaled to any case of similar geometry. The relationship between the results of this theory and those to be expected in the case of an insulator of finite thickness is discussed.

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